QUANTUM CYLINDRIC SET ALGEBRAS

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In memory of Dick Greechie

Abstract. Quantum cylindric algebras were introduced by the author as a generalization of their classical counterpart. Primary examples based on Weaver's treatment of the tensor power of a Hilbert space with diagonals given by symmetric tensor products are called full quantum cylindric set algebras. We generalize a result of Sudkamp in the classical setting and axiomatize n-dimensional full quantum cylindric set algebras.

1. Introduction

Cylindric algebras were introduced by Tarski and his students Chin and Thompson from 1948-52 (see [5] for details) to provide an algebraic treatment of first-order logic. For a cardinal κ, a κ-dimensional cylindric algebra consists of a Boolean algebra B with a family c_α $(\alpha \in \kappa)$ of unary operations and for each pair $\alpha, \beta \in \kappa$ a constant $d_{\alpha,\beta}$, all subject to certain equational axioms. The idea is that κ is the cardinality of a set of variables, each c_α expresses universal quantification over that variable, and the $d_{\alpha,\beta}$ expresses equality of those variables. There is an extensive literature on cylindric algebras [3, 4] and the related monadic algebras developed by Halmos [1].

A primary example of a κ-dimensional cylindric algebra is obtained by taking a set X, considering the Boolean algebra of all subsets of X^κ. For $\alpha \in \kappa$, the *cylindrification* c_α of a subset $A \subseteq X^\kappa$ is the set of all choice functions $\sigma \in X^\kappa$ that agree with a choice function in A except possibly at the α coordinate. The diagonal $d_{\alpha,\beta}$ is the set of all choice functions $\sigma \in X^\kappa$ with $\sigma(\alpha) = \sigma(\beta)$. This is called a *full cylindric set algebra*. The term *cylindric set algebra* is used for any subalgebra of a full cylindric set algebra.

In [9] Sudcamp gave a characterization of n-dimensional full cylindric set algebras. Let B be an n-dimensional cylindric algebra. Sudkamp defines B to be *strong* if its underlying Boolean algebra is complete and atomic, and using At for the atoms of B and c^i for the composite of all cylindrifications c_j where $j \neq i$, satisfies

(T_1) if $p \neq 0$, then $c_1 \cdots c_n\, p = 1$,

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(T$_2$) if $x_1, \dots, x_n \in \mathsf{At}$ then $\bigwedge_1^n c^i x_i \in \mathsf{At}$.

Sudkamp shows that an n-dimensional cylindric algebra is isomorphic to a full cylindric set algebra if and only if it is strong.

The notion of a *κ-dimensional quantum cylindric algebra* was introduced in [2]. This consists of an orthomodular lattice (abbrev.: OML) with a family of unary operations c_α ($\alpha \in \kappa$) and for each $\alpha, \beta \in \kappa$ a constant $d_{\alpha,\beta}$ satisfying a subset of the axioms used to define cylindric algebras. While a range of examples were given from subfactor theory, the relevant issue here is that of a *full n-dimensional quantum cylindric set algebra*. Such is obtained by taking a tensor power $\mathcal{H}^{\bigotimes n}$ of a Hilbert space $\mathcal{H}$, considering its OML L of closed subspaces, defining operations c_i via certain complete subalgebras of L, and making diagonals $d_{i,j}$ from obvious variants of the symmetric tensor product. This is obtained from a path originally suggested in Weaver [11] and somewhat related to work of Kornell on quantum sets [7].

The purpose of this note is to provide a result analogous to that of Sudkamp for quantum cylindric algebras. We provide axioms to define a *strong n-dimensional quantum cylindric algebra* and then show that an n-dimensional quantum cylindric algebra is strong if and only if it is isomorphic to a full n-dimensional quantum cylindric set algebra.

This paper is arranged as follows. The second section has the basics of quantum cylindric algebras and defines when a quantum cylindric algebra is strong, both in the diagonal-free setting and in the diagonal setting. The third section describes full n-dimensional quantum cylindric set algebras, both in the diagonal-free and diagonal setting, and shows that they are strong. The fourth section shows that any strong diagonal-free n-dimensional quantum cylindric algebra is isomorphic to a diagonal-free full n-dimensional quantum cylindric set algebra. The fifth section establishes the corresponding result in the setting with diagonals, albeit with a familiar assumption ruling out an I_2-factor. The sixth section contains remarks on axiomatics.

2. Quantum cylindric algebras

We begin with the basic definition of a finite-dimensional quantum cylindric algebra (we don't have need consider higher dimension here). We note that this was called a *weak quantum cylindric algebra* in [2, Def. 5.14], but the current terminology seems more appropriate.

Definition 2.1. *For a natural number n, a diagonal-free n-dimensional quantum cylindric algebra is an* OML *L with unary operations c_i for $i \leq n$ so that for $i, j, \leq n$*

(C$_1$) $c_i 0 = 0$,

(C$_2$) $p \leq c_i p$,

(C$_3$) $c_i(p \vee q) = c_i p \vee c_i q$,
(C$_4$) $c_i c_i p = c_i p$,
(C$_5$) $c_i(c_i p)^\perp = (c_i p)^\perp$,
(C$_6$) $c_i c_j x = c_j c_i x$.

The system is an n-dimensional quantum cylindric algebra if it additionally has constants $d_{i,j}$ for $i, j, \leq n$ such that for any $i, j, k \leq n$ we have

(C$_7$) *$d_{i,j} = d_{j,i}$ and $d_{i,i} = 1$,*
(C$_8$) *if $j \neq i, k$ then $d_{i,k} = c_j(d_{i,j} \wedge d_{j,k})$.*

Remark 2.2. The axioms for quantum cylindric algebras are a weakening of those for cylindric algebras in two ways. First, the underlying structure is assumed only to be an OML rather than a Boolean algebra. Also, an axiom for cylindric algebras is simply omitted. This axiom says that for $i \neq j$, $c_i(d_{i,j} \wedge x) \wedge c_j(d_{i,j} \wedge x') = 0$ and provides that the operation $S^i_j x = c_i(d_{i,j} \wedge x)$ is an endomorphism of the underlying Boolean algebra and can serve as a substitution operation. This axiom fails in interesting examples of quantum cylindric algebras and no replacement has been found. For these reasons, the quantum cylindric algebra axioms are significantly weaker than those for cylindric algebras.

A unary operation c is called a quantifier if it satisfies (C$_1$) – (C$_5$). These conditions say that it is a closure operation where the orthocomplement of a closed element is closed. Conditions (C$_4$) and (C$_6$) say that compositions of the operations c_i ($i \leq n$) of a diagonal-free quantum cylindric algebra form a commutative, idempotent submonoid of the endomorphism monoid. Here the empty composite is considered to be the identity.

Lemma 2.3. *The composition of two commuting quantifiers is a quantifier.*

Proof. The composition of two operations that preserve finite joins, are order preserving, and are increasing is again such; and if the two operations are idempotent and commute, then their composition is idempotent. For condition (C$_5$), suppose that c_1, c_2 are commuting quantifiers. Then $c_1c_2(c_1c_2p)^\perp = c_1c_2(c_2c_1p)^\perp = c_1(c_2c_1p)^\perp = c_1(c_1c_2p)^\perp = (c_1c_2p)^\perp$. □

Definition 2.4. *In an n-dimensional diagonal-free quantum cylindric algebra set for $i \leq n$*

$$
\begin{aligned}
c^i &= c_1 \cdots c_{i-1} c_{i+1} \cdots c_n, \\
Q_i &= \textit{the range of the quantifier } c_i, \\
Q^i &= \textit{the range of the quantifier } c^i, \\
\mathsf{At} &= \textit{the set of atoms of } L, \\
\mathsf{At}_i &= \textit{the set of atoms of } Q_i, \\
\mathsf{At}^i &= \textit{the set of atoms of } Q^i, \\
\mathsf{At}_u &= \{x \in \mathsf{At} : x = \textstyle\bigwedge_1^n c^i x\}.
\end{aligned}
$$

In an n-dimensional quantum cylindric algebra, set

$$
\begin{aligned}
d &= \bigwedge\{d_{i,j} : i, j \leq n\}, \\
\mathsf{At}_d &= \{x \in \mathsf{At}_u : x \leq d\}.
\end{aligned}
$$

We refer to d as the generalized diagonal*.*

Remark 2.5. Consideration of the range of a quantifier is very natural. The range of a quantifier on a complete OML L is a *complete subalgebra*, meaning that it is a sub-OML and is closed under all joins and meets as taken in L. Conversely, any complete subalgebra of a complete OML gives a quantifier. These easy facts are found in [2].

Lemma 2.6. *If L is an n-dimensional diagonal-free quantum cylindric algebra, then for each $i \leq n$ we have $Q^i = \bigcap\{Q_j : j \neq i\}$.*

Proof. Without loss of generality we assume $i = 1$ so $c^i = c_2 \cdots c_n$. Suppose $p \in Q^i$. Then $c^i p = p$ so $p = c_2 \cdots c_n p$ and this gives $p = c_j p$ for each $j \neq i$, hence $p \in Q_j$ for each $j \neq i$, giving $p \in \bigcap\{Q_j : j \neq i\}$. Conversely, if $p \in \bigcap\{Q_j : j \neq i\}$ then $p \in Q_j$ for each $j \neq i$, giving $c_j p = p$ for each $j \neq i$. Then $c^i p = c_2 \cdots c_n \, p = p$, giving $p \in Q^i$. $\square$

We next give definitions of strong for n-dimensional diagonal-free quantum cylindric algebras and for n-dimensional quantum cylindric algebras. We recall that elements x, y of an OML L *commute* if they generate a Boolean subalgebra of L and for $S \subseteq L$ we write $C(S)$ for the set of all elements of L that commute with each element in S. See [6] for details.

Definition 2.7. *An n-dimensional diagonal-free quantum cylindric algebra is* strong *if L is isomorphic to the projection lattice of a Hilbert space and for any $i \leq n$ and $p \neq 0$*

- (S_1) $c_1 \cdots c_n p = 1$,
- (S_2) $C(Q^i) = Q_i$,
- (S_3) *there is $x \in \mathsf{At}_u$ with $x \leq c^i p$,*
- (S_4) *if $x, y \in \mathsf{At}_u$, then $x \leq c^i y$ implies $y \leq c^i x$.*

An n-dimensional quantum cylindric algebra is strong *if L is isomorphic to the projection lattice of a Hilbert space and for any $i,j \leq n$ and $p \neq 0$*

(S_1) $c_1 \cdots c_n p = 1$,
(S_2) $C(Q^i) = Q_i$,
(S_3^d) *there is $x \in \mathsf{At}_d$ with $x \leq c^i p$,*
(S_4^d) *if $x, y \in \mathsf{At}_d$, then $x \leq c^i y$ implies $x = y$,*
(S_5) *if $x, y \in \mathsf{At}_d$, then $x \perp y$ implies $c^i x \perp c^i y$,*
(S_6) $d_{i,j} = \bigvee\{c^i x \wedge c^j x : x \in \mathsf{At}_d\}$.

3. Quantum cylindric set algebras

In this section we recall from [2] the definition of full n-dimensional quantum set algebras, both with and without diagonals, and then show that they are strong. We use the following notations. For a Hilbert space $\mathcal{H}$, let $P(\mathcal{H})$ be its self-adjoint projections and $\mathcal{C}(\mathcal{H})$ be its closed subspaces. It is well known that these are isomorphic OMLs. For $v \in \mathcal{H}$ we use $\langle v\rangle$ for its span and for $S \subseteq \mathcal{H}$ we use $\langle S\rangle$ for the closure of its span.

Definition 3.1. *For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let*

$$\mathcal{H} \otimes \mathcal{C}(\mathcal{K}) = \{\mathcal{H} \otimes A : A \in \mathcal{C}(\mathcal{K})\}.$$

This is a complete sub-OML of $\mathcal{C}(\mathcal{H} \otimes \mathcal{K})$ and therefore yields a quantifier $\exists_{\mathcal{H}}$ on $\mathcal{C}(\mathcal{H} \otimes \mathcal{K})$ where $\exists_{\mathcal{H}} S$ is the least member of $\mathcal{H} \otimes \mathcal{C}(\mathcal{K})$ that lies above S.

In the following, we use associativity of finite tensor products.

Definition 3.2. *For Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ and $i \leq n$ let $\exists_{\mathcal{H}_i}$ be the quantifier associated to the complete sub-OML of $\mathcal{C}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ given by*

$$\mathcal{H}_i \otimes \mathcal{C}(\textstyle\bigotimes_{j \neq i} \mathcal{H}_j)$$

*Then the OML of closed subspaces of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with quantifiers $\exists_{\mathcal{H}_1}, \ldots \exists_{\mathcal{H}_n}$ is the n-*dimensional full diagonal-free quantum cylindric set algebra *associated with $\mathcal{H}_1, \ldots, \mathcal{H}_n$.*

An explicit description of the quantifier $\exists_{\mathcal{H}}$ on $\mathcal{C}(\mathcal{H} \otimes \mathcal{K})$ is given in [2, Sec. 6]. Let $(a_i)_I$ be an ONB of $\mathcal{H}$. Each $v \in \mathcal{H} \otimes \mathcal{K}$ can be uniquely expressed as $\sum_I a_i \otimes v_i$ for some family $(v_i)_I \in \mathcal{K}$. For $S \subseteq \mathcal{H} \otimes \mathcal{K}$ set

$$S^{\mathcal{H}} = \langle v_i : v \in S \text{ and } i \in I\rangle$$

Then by [2, Prop. 6.7], if $S \in \mathcal{C}(\mathcal{H} \otimes \mathcal{K})$, we have $\exists_{\mathcal{H}} S = \mathcal{H} \otimes S^{\mathcal{H}}$.

Proposition 3.3. *Let L be the n-dimensional diagonal-free quantum cylindric set algebra associated to $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Then for $i \leq n$ we have*

(1) $Q_i = \mathcal{H}_i \otimes \mathcal{C}(\bigotimes_{j \neq i} \mathcal{H}_j)$,
(2) $Q^i = \mathcal{C}(\mathcal{H}_i) \otimes \bigotimes_{j \neq i} \mathcal{H}_j$,

(3) $\mathsf{At}_i = \{\mathcal{H}_i \otimes \langle v\rangle : 0 \neq v \in \bigotimes_{j\neq i}\mathcal{H}_j\}$,
(4) $\mathsf{At}^i = \{\langle a\rangle \otimes \bigotimes_{j\neq i}\mathcal{H}_j : 0 \neq a \in \mathcal{H}_i\}$,
(5) $\mathsf{At}_u = \{\langle a_1\rangle \otimes \cdots \otimes \langle a_n\rangle : 0 \neq a_i \in \mathcal{H}_i$ *for* $i \leq n\}$.

Proof. (1) This is by definition. (2) By definition, Q^i is the range of the quantifier c^i that is the composite of all the quantifiers $\exists_{\mathcal{H}_j}$ where $j \neq i$. Using [2, Prop. 6.9] and a simple induction, this is the quantifier $\exists_{\bigotimes_{j\neq i}\mathcal{H}_j}$. The range of this quantifier is $\mathcal{C}(\mathcal{H}_i) \otimes \bigotimes_{j\neq i}\mathcal{H}_j$. (3) and (4) are trivial from (1) and (2). (5) Let $x = \langle a_1\rangle \otimes \cdots \otimes \langle a_n\rangle$. Clearly x is an atom of L. Since $\exists_{\mathcal{H}_1} x = \mathcal{H}_1 \otimes \langle a_2\rangle \cdots \otimes \langle a_n\rangle$, ..., $\exists_{\mathcal{H}_n} x = \langle a_1\rangle \otimes \cdots \otimes \langle a_{n-1}\rangle \otimes \mathcal{H}_n$, we have that $\bigwedge_1^n \exists_{\mathcal{H}_i} x = x$, giving $x \in \mathsf{At}_u$. Conversely, suppose $x \in \mathsf{At}_u$. Then $\exists_{\mathcal{H}_i} x = A_i \otimes \bigotimes_{j\neq i}\mathcal{H}_i$ for some $A_i \in \mathcal{C}(\mathcal{H}_i)$ and so $\bigwedge_1^n \exists_{\mathcal{H}_i} x = A_1 \otimes \cdots \otimes A_n$. Since this latter expression is equal to the atom x, it must be that $A_i = \langle a_i\rangle$ for some non-zero $a_i \in \mathcal{H}_i$, hence $x = \langle a_1\rangle \otimes \cdots \otimes \langle a_n\rangle$. $\square$

Remark 3.4. Making use of the isomorphism between projections and closed subspaces, the subalgebra $\mathcal{H}_i \otimes \mathcal{C}(\bigotimes_{j\neq i}\mathcal{H}_j)$ can be realized as $1_{\mathcal{H}_i} \otimes \mathcal{P}(\bigotimes_{j\neq i}\mathcal{H}_j)$ where this is the collection of all projection operators that are obtained as the tensor product of the identity projection on $\mathcal{H}_i$ and a projection on the tensor product of the other factors.

Proposition 3.5. *For Hilbert spaces* $\mathcal{H}_1, \ldots, \mathcal{H}_n$*, the associated* n*-dimensional diagonal-free full quantum cylindric algebra is a strong* n*-dimensional diagonal-free quantum cylindric algebra.*

Proof. That this structure is an n-dimensional diagonal-free cylindric algebra is established in [2, Th, 6.10]. We show that it is strong. For (S_1) note that as in the proof of Lemma 2.6 the range of the quantifier $c_1 \cdots c_n$ is $Q_1 \cap \cdots \cap Q_n$. By known properties of tensor products this consists of only $\bigotimes_1^n \mathcal{H}_i$ and the zero subspace (the bounds of the OML) and the result follows. For (S_2) it is well known that $C(\mathcal{C}(\mathcal{H}) \otimes \mathcal{K}) = \mathcal{H} \otimes \mathcal{C}(\mathcal{K})$. The result then follows from the descriptions of Q_i and Q^i given in Proposition 3.3. For (S_3) we have that $c^i p$ is a non-zero element of Q^i and so is equal to $A \otimes \bigotimes_{j\neq i}\mathcal{H}_j$ for some non-zero closed subspace A of $\mathcal{H}_i$. For $j \leq n$ choose non-zero $a_j \in \mathcal{H}_j$ with $a_i \in A$ and set $x = \langle a_1\rangle \otimes \cdots \otimes \langle a_n\rangle$. Then $x \leq c^i p$ and by Proposition 3.3 $x \in \mathsf{At}_u$. For (S_4) assume $x, y \in \mathsf{At}_u$. By Proposition 3.3 for $i \leq n$ there are non-zero $a_i, b_i \in \mathcal{H}_i$ with $x = \langle a_1\rangle \otimes \cdots \otimes \langle a_n\rangle$ and $y = \langle b_1\rangle \otimes \cdots \otimes \langle b_n\rangle$. Since $c^i y$ is least in Q^i above y, we have $c^i y = \langle b_i\rangle \otimes \bigotimes_{j\neq i}\mathcal{H}_j$. Thus $x \leq c^i y$ iff $\langle a_i\rangle = \langle b_i\rangle$, and by symmetry this occurs iff $y \leq c^i x$. $\square$

For diagonals, which express a type of equality, we need a tensor power $\mathcal{H}^{\otimes n}$. The treatment is based on the idea from [11]. For $F \subseteq \{1, \ldots, n\}$ the diagonal D_F is the closed subspace of $\mathcal{H}^{\otimes n}$ that is the symmetric tensor product of those factors $\mathcal{H}_i$ with $i \in F$ tensored with the tensor product of those factors $\mathcal{H}_i$ with $i \notin F$. A more explicit description in given in [2, Sec. 6] based on a well-known description of the symmetric tensor product in terms of an orthonormal basis (abbrev.: ONB). If $(a_i)_I$ is an ONB of $\mathcal{H}$ then each $v \in \mathcal{H}^{\otimes n}$ has a unique representation

$$v = \sum_{\alpha \in I^n} \lambda_\alpha a_{\alpha_1} \otimes \cdots \otimes a_{\alpha_n}$$

Note that if $\sigma \in \mathsf{Perm}(n)$ and $\alpha \in I^n$, then the composite $\alpha\sigma = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ is in I^n.

Definition 3.6. *Let $\mathcal{H}$ be a Hilbert space with* ONB $(a_i)_I$. *Then for $F \subseteq \{1, \ldots, n\}$ let*

$$D_F = \left\{ \sum_{\alpha \in I^n} \lambda_\alpha a_{\alpha_1} \otimes \cdots \otimes a_{\alpha_n} : \lambda_\alpha = \lambda_{\alpha\sigma} \textit{ for all } \sigma \in \mathsf{Perm}(n) \right\}$$

We use $D_{i,j}$ for $D_{\{i,j\}}$ and D for $D_{\{1,\ldots,n\}}$.

Yet another description of the diagonals D_F is obtained from the well-known result that an n-fold symmetric tensor product $\mathcal{H} \otimes_s \cdots \otimes_s \mathcal{H}$ is the closure of the span of all vectors of the form $a \otimes \cdots \otimes a$ where $a \in \mathcal{H}$.

Lemma 3.7. *For $F \subseteq \{1, \ldots, n\}$ of cardinality k,*

$$D_F = \left\langle \underbrace{a \otimes \cdots \otimes a}_{k \textit{ times}} \otimes v : a \in \mathcal{H} \textit{ and } v \in \bigotimes_{i \notin F} \mathcal{H} \right\rangle$$

Definition 3.8. *Let $\mathcal{H}$ be a Hilbert space and n be a natural number. The n-*dimensional full quantum cylindric set algebra *over $\mathcal{H}$ is the n-dimensional diagonal-free quantum cylindric set algebra associated with $\mathcal{H}^{\otimes n}$ equipped with diagonals $D_{i,j}$ for $i, j \leq n$.*

We conclude this section with the promised result showing that full quantum cylindric algebras are strong.

Proposition 3.9. *For a Hilbert space $\mathcal{H}$ and $n \in \mathbb{N}$, the n-dimensional full quantum cylindric set algebra over $\mathcal{H}$ is a strong n-dimensional quantum cylindric algebra. Further, in it $\mathsf{At}_d = \{\langle a \rangle \otimes \cdots \otimes \langle a \rangle : a \in \mathcal{H}\}$.*

Proof. That this structure is an n-dimensional diagonal-free cylindric algebra is established in [2, Sec. 6]. (Note that this was called a weak quantum cylindric algebra in [2]). We show here that this quantum cylindric algebra is strong. Since the reduct is a full diagonal-free quantum cylindric set algebra, (S_1) and (S_2) hold. Before verifying the other conditions, which involve At_d, we establish the stated description of this set.

By definition At_d is those elements of At_u that lie beneath the generalized diagonal D, hence that lie in the antisymmetric tensor product. By Proposition 3.3, elements of At_u are those of the form $\langle a_1\rangle \otimes \cdots \otimes \langle a_n\rangle$ where $a_1, \ldots, a_n \in \mathcal{H}$. So each $\langle a\rangle \otimes \cdots \otimes \langle a\rangle$ for $a \in \mathcal{H}$ belongs to At_d. But if $\langle a_1\rangle \otimes \cdots \otimes \langle a_n\rangle$ belongs to the symmetric tensor product, then from one of the many descriptions of the symmetric tensor product $a_1 \otimes \cdots \otimes a_n$ is fixed under permutations of indices, so $a_1, \ldots, a_n$ are scalar multiples of one another. So At_d is as described.

Using this description of At_d, the proof of (S_3^d) involves only obvious modifications to the proof of (S_3) from Proposition 3.5. For (S_4^d) suppose $x, y \in \mathsf{At}_d$. Then there are $0 \neq a, b \in \mathcal{H}$ with $x = \langle a\rangle \otimes \cdots \otimes \langle a\rangle$ and $y = \langle b\rangle \otimes \cdots \otimes \langle b\rangle$. Since $c^i x = \langle a\rangle \otimes \bigotimes_{j\neq i} \mathcal{H}$ and $c^i y = \langle b\rangle \otimes \bigotimes_{j\neq i} \mathcal{H}$, if $c^i x = c^i y$ then $\langle a\rangle = \langle b\rangle$, hence $x = y$. For (S_5) let $x, y \in \mathsf{At}_d$ as above and assume $x \perp y$. Then $\langle a\rangle \perp \langle b\rangle$ and it follows that $c^i x \perp c^i y$. For (S_6) if $x = \langle a\rangle \otimes \cdots \otimes \langle a\rangle$ belongs to At_d, then $c^i x \wedge c^j x = \langle a\rangle \otimes \langle a\rangle \otimes \bigotimes_{k\neq i,j} \mathcal{H}_k$. By Lemma 3.7 such elements generate $D_{i,j}$ as a closed subspace. □

4. Strong implies set for diagonal-free

In this section, we establish that a strong, n-dimensional diagonal-free quantum cylindric algebra is an n-dimensional diagonal-free full quantum cylindric set algebra. The subsequent section deals with diagonals. To avoid repetitious work, it is useful for the reader to note that the proofs of this section requiring (S_3) or (S_4) will hold equally well when replaced by (S_3^d) and (S_4^d) in the following section. Throughout this section we assume that L is a strong n-dimensional diagonal-free quantum cylindric algebra whose underlying lattice is the projection lattice of the Hilbert space $\mathcal{H}$.

Proposition 4.1. *For each $i \leq n$, $Q_i \cap Q^i = \{0, 1\}$.*

Proof. By Lemma 2.6 $Q^i = \bigcap\{Q_j : j \neq i\}$, so $Q_i \cap Q^i = \bigcap\{Q_i : i \leq n\}$. Surely $0, 1$ belong to this intersection. If p belongs to this intersection, then $p = c_i p$ for each $i \leq n$, hence $p = c_1 \cdots c_n\, p$. By (S_1) if $p \neq 0$ then $p = 1$. □

We use B for the set of all bounded operators of $\mathcal{H}$, and for $A \subseteq B$ we use A' for the set of operators that commute with each operator in A. The set A' is called the *commutant* of A. If A is closed under adjoints, expressed by saying that A is *self-adjoint*, the *double commutant* A'' is the unital von Neumann subalgebra of B generated by A. We follow Murray and von Neumann in calling unital von Neumann subalgebras of B *subrings* of B. It is well-known [6] that for projections, commuting in the operator sense is equivalent to commuting in the lattice-theoretic sense.

Lemma 4.2. *For $P \subseteq L$ we have*

$$C(C(P)) = P'' \cap L.$$

Thus, $P = C(C(P))$ iff P is the set of projections of a subring of B.

Proof. "$\supseteq$". Suppose $p \in P'' \cap L$. Then p is a projection that commutes with every operator in P'. In particular, p commutes with every projection in P', hence with every member of $C(P)$. So $p \in C(C(P))$.

"$\subseteq$". Note that P is a subset of the bounded operators B of $\mathcal{H}$ that is self-adjoint and contains the identity, and therefore the commutant P' has these same properties. Further, since $P' = P'''$ by basic principles, we have that P' is a von Neumann algebra.

Suppose a is a self-adjoint element of P'. Then the abelian von Neumann algebra A that is generated by a is contained in P'. The projections providing the spectral representation of a are all in A, and hence in P'. But these are projections and therefore belong to $C(P)$.

If $p \in C(C(P))$ and a is a self-adjoint element of P' we have that p commutes with all of the spectral projections for a since these all belong to $C(P)$, and it follows that p commutes with a. Since every element in the von Neumann algebra P' is a linear combination of self-adjoint elements in P', thus p commutes with every element of P', hence $p \in P'' \cap L$.

To see the final remark, suppose $P = C(C(P))$. Then $P = P'' \cap L$. But P is self-adjoint since it consists of projections. Thus P'' is a unital von Neumann subalgebra, so P is the projections of a unital von Neumann subalgebra of B. Conversely, if M is a unital von Neumann subalgebra of B and P is its set of projections, then it is known that M is generated as a von Neumann algebra by P, so $M = P''$. Thus $P = P'' \cap L$, giving $P = C(C(P))$. □

Definition 4.3. *Let $M_i = (Q_i)''$ and $M^i = (Q^i)''$.*

Note that since Q_i and Q^i are self-adjoint, M_i and M^i are subrings of B.

Proposition 4.4. *For $i \leq n$*

(1) $Q_i = M_i \cap L$,
(2) $Q^i = M^i \cap L$,
(3) $M^i = \bigcap\{M_j : j \neq i\}$,
(4) $M_i = (M^i)'$,
(5) $M^i = (M_i)'$.

Proof. (1) By (S_2) we have $Q_i = C(Q^i)$. This gives $C(C(Q_i)) = Q_i$. The result follows by Lemma 4.2. (2) By Lemma 2.6 we have $Q^i = \bigcap\{Q_j : j \neq i\}$, it then follows from (1) that $Q^i = \bigcap\{M_j : j \neq i\} \cap L$. Since the intersection of subrings is a subring, we have that Q^i is the projections of a subring, so by Lemma 4.2 we have $Q^i = (Q^i)'' \cap L = M^i \cap L$. (3) The subrings M^i

and $\bigcap\{M_j : j \neq i\}$ have the same projections and each subring is generated by its projections, so they are equal. (4) By (S_2) $Q_i = C(Q^i)$. So by (1) we have $M_i \cap L = (Q^i)' \cap L$. Since both M_i and $(Q^i)'$ are subrings, and subrings are determined by their projections, this gives $M_i = (Q^i)'$. But $(Q^i)' = (Q^i)''' = (M^i)'$. (5) Since $M_i = (M^i)'$, we have $(M_i)' = (M^i)''$, and since M^i is a subring, $(M^i)'' = M^i$. □

The following is from [8, Def 3.1.1] where the notation $R(N_1, \ldots, N_n)$ was used for the subring generated by the union of $N_1, \ldots, N_n$.

Definition 4.5. *A family $N_1, \ldots, N_n$ of subrings of B are a factorization if $N_i \subseteq N_j'$ for each $i \neq j$ and $R(N_1, \ldots, N_n) = B$.*

A *factor* [8, Def. 3.1.2] is a subring N with trivial intersection with its commutant. Several basic facts are found in [8, p. 28]. A subring N is a factor iff N, N' is a factorization; each member of a factorization is a factor; and if $N_1, \ldots, N_n$ is a factorization, then for each i the pair N_i, $R(\{N_j : j \neq i\})$ is factorization. In [8, Def. 3.1.3] a factorization N_1, N_2 is called *coupled* if $N_2 = N_1'$. Since $N_1'' = N_1$ for any subring, in a coupled factor $N_2' = N_1$. We call a factorization $N_1, \ldots, N_n$ coupled if each $N_i, R(\{N_j : j \neq i\})$ is coupled.

Proposition 4.6. *$M^1, \ldots, M^n$ is a coupled factorization of $\mathcal{H}$.*

Proof. By Proposition 4.4.4 $(M^j)' = M_j$. Since $M^i = \bigcap\{M_j : j \neq i\}$ we have $M^i \subseteq (M^j)'$ for each $j \neq i$. Having $R(M^1, \ldots, M^n) = B$ is equivalent to having $(M^1)' \cap \cdots \cap (M^n)' = \mathbb{C}I$, hence to having $M_1 \cap \cdots \cap M_n = \mathbb{C}I$. But $M_2 \cap \cdots \cap M_n = (M_1)'$. By Propositions 4.1 and 4.2 $M_1 \cap M^1 \cap L = \{0, 1\}$, hence $M_1 \cap M^1 = \mathbb{C}I$ as required. Since the subring generated by $\{M^j : j \neq i\}$ is equal to $(\bigcap\{(M^j)' : j \neq i\})' = (M^i)'$, this factorization is coupled. □

Thus far, we have used only (S_1) and (S_2). We make use of the remaining conditions (S_3) and (S_4) in the following.

Proposition 4.7. *Q_i, Q^i are atomic and the atoms of Q^i are $\{c^i x : x \in \mathsf{At}_u\}$.*

Proof. Suppose $x \in \mathsf{At}_u$. If $0 \neq p \in Q^i$ and $p \leq c^i x$, by (S_3) there is $y \in \mathsf{At}_u$ with $y \leq c^i p = p$, hence with $y \leq c^i y \leq p$. Then $y \leq p \leq c^i x$ gives by (S_4) that $x \leq c^i y$. So $x \leq p$. This implies $c^i x \leq p$, giving $p = c^i x$. So $c^i x$ is an atom of Q^i. By (S_3) each $0 \neq p \in Q^i$ lies above some $x \in \mathsf{At}_u$, so lies above the atom $c^i x$ of Q^i. It follows that Q^i is atomic and its atoms are as described. Thus M^i is a factor that has minimal projections. So by [8, Lem. 5.3.1] the same is true of $(M^i)' = M_i$. By Lemma 4.4.4 the projections of M_i are Q_i. Since a factor with a minimal projection has an atomic projection lattice, the result follows. □

Remark 4.8. The reader should consider the proof of Proposition 4.7 in the setting with diagonals and note that an analogous conclusion is obtained using (S_3^d) and (S_4^d) in place of (S_3) and (S_4). Specifically, if (S_3^d) and (S_4^d) hold then Q_i, Q^i are atomic and the atoms of Q^i are $\{c^i x : x \in \mathsf{At}_d\}$.

A factorization $N_1, \ldots, N_n$ is a *direct factorization* [8, Def. 3.2.1] if $\mathcal{H}$ is isomorphic to $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ in such a way that N_i becomes the subring denoted there as $B^{(i)}$ and which is given by [8, Def. 2.3.1] $1 \otimes \cdots \otimes B_i \otimes \cdots \otimes 1$, which is the collection of all operators of the form $1 \otimes \cdots \otimes A_i \otimes \cdots \otimes 1$ where A_i is a bounded operator of $\mathcal{H}_i$.

Proposition 4.9. *$M^1, \ldots, M^n$ is a direct factorization.*

Proof. By Proposition 4.6, $M^1, \ldots, M^n$ is a factorization, and by Proposition 4.7 each factor M^i has a minimal projection. Thus by [8, Thm. IV, p.40] each factor M^i is direct, and so by [8, Lem. 5.4.1] the factorization $M^1, \ldots, M^n$ is direct. □

Theorem 4.10. *Each strong n-dimensional diagonal-free quantum cylindric algebra is an n-dimensional full diagonal-free quantum cylindric set algebra.*

Proof. By Proposition 4.9 the factorization $M^1, \ldots, M^n$ is direct. So $\mathcal{H}$ is isomorphic to $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ in such a way that M^i becomes $B^{(i)} = 1 \otimes \cdots \otimes B_i \otimes \cdots \otimes 1$. Basic properties of tensor products give $(M^i)' = 1_{\mathcal{H}_i} \otimes B(\bigotimes_{j\neq i} \mathcal{H}_j)$. Since $(M^i)' = M_i$, it follows that $Q_i = M_i \cap L = 1_{\mathcal{H}_i} \otimes \mathcal{P}(\bigotimes_{j\neq i} \mathcal{H}_j)$. Then since Q_i is the complete subalgebra of L associated to the quantifier c_i and this subalgebra is isomorphic to $\mathcal{H}_i \otimes \mathcal{C}(\bigotimes_{j\neq i} \mathcal{H}_j)$, the result follows from Definition 3.2. □

5. Strong implies set with diagonals

In this section we show that under mild conditions, a strong n-dimensional quantum cylindric algebra is isomorphic to an n-dimensional full quantum cylindric set algebra. We begin with the following.

Proposition 5.1. *The diagonal-free reduct of a strong n-dimensional quantum cylindric algebra is a diagonal-free full quantum cylindric set algebra.*

Proof. The diagonal-free reduct satisfies (S_1) and (S_2). It also satisfies (S_3^d) and (S_4^d), which according to Remark 4.8 is enough to obtain a slightly modified form of Proposition 4.7. This modified form of Proposition 4.7 gives minimal projections that are needed for the proof of Theorem 4.10. □

In the following, we assume that L is a strong n-dimensional quantum cylindric algebra. In view of Proposition 5.1, the diagonal-free reduct of L is a diagonal-free full quantum cylindric set algebra. We make use of the notation and results of the previous section. In particular, there are Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ so that L is the projection lattice of $\bigotimes_1^n \mathcal{H}_i$. For $i \leq n$, the ranges of the quantifiers c_i, c^i are Q_i, Q^i and the descriptions of Q_i, Q^i and of the atom sets $\mathsf{At}_i, \mathsf{At}^i$ and At_u are as in Proposition 3.3.

Definition 5.2. *An orthogonality space* $(X, \perp)$ *is a set* X *with a symmetric binary relation* $\perp$ *on it. An isomorphism between orthogonality spaces is a bijection* ϕ *with* $p \perp q$ *iff* $\phi p \perp \phi q$.

Any subset of L can be considered as an orthogonality space where the orthogonality relation is the restriction of orthogonality in the OML L, namely with $p \perp q$ iff $p \leq q^\perp$. In particular, we consider At^i and At_d as orthogonality spaces.

Proposition 5.3. $c^i : \mathsf{At}_d \to \mathsf{At}^i$ *is an isomorphism of orthogonality spaces.*

Proof. As discussed in Remark 4.8, the proof of Proposition 4.7 carries over with minor modification, using (S_3^d) and (S_4^d) in place of (S_3) and (S_4), to give that the atoms At^i of Q^i are $\{c^i x : x \in \mathsf{At}_d\}$. So $c^i : \mathsf{At}_d \to \mathsf{At}^i$ is well defined and onto. Axiom (S_4) gives that c^i is one-one, and then (S_5) provides that it is an orthogonality space isomorphism. $\square$

Definition 5.4. *For a Hilbert space* $\mathcal{H}$*, let* $\mathsf{At}_\mathcal{H}$ *be the orthogonality space formed from the atoms of* $\mathcal{C}(\mathcal{H})$*.*

Elements of $\mathsf{At}_{\mathcal{H}_i}$ are those of the form $\langle a_i \rangle$ where $a_i \in \mathcal{H}_i$ is non-zero. By Proposition 3.3, $Q^i = \mathcal{C}(\mathcal{H}_i) \otimes \bigotimes_{j \neq i} \mathcal{H}_j$, so elements of At^i are those of the form $\langle a_i \rangle \otimes \bigotimes_{j \neq i} \mathcal{H}_j$. So there are mutually inverse orthogonality space isomorphisms $\mu^i : \mathsf{At}^i \to \mathsf{At}_{\mathcal{H}_i}$ and $\nu^i : \mathsf{At}_{\mathcal{H}_i} \to \mathsf{At}^i$ that interchange $\langle a_i \rangle$ and $\langle a_i \rangle \otimes \bigotimes_{j \neq i} \mathcal{H}_j$. Since $\mathsf{At}_d \subseteq \mathsf{At}_u$, each element of At_d is of the form $\langle a_1 \rangle \otimes \cdots \otimes \langle a_n \rangle$ for some family $\langle a_1 \rangle \in \mathsf{At}_{\mathcal{H}_1}, \ldots, \langle a_n \rangle \in \mathsf{At}_{\mathcal{H}_n}$. The isomorphism $c^i : \mathsf{At}_d \to \mathsf{At}^i$ takes $\langle a_1 \rangle \otimes \cdots \otimes \langle a_n \rangle$ to $\langle a_i \rangle \otimes \bigotimes_{j \neq i} \mathcal{H}_j$. The inverse of c^i, which we write d^i, takes $\langle a_i \rangle \otimes \bigotimes_{j \neq i} \mathcal{H}_j$ to the unique member of At_d that has $\langle a_i \rangle$ as its i^{th} tensor factor. The composite $\gamma^{i,j} : \mathsf{At}_{\mathcal{H}_i} \to \mathsf{At}_{\mathcal{H}_j}$ is an orthogonality space isomorphism. It takes $\langle a_i \rangle$ to $\langle a_j \rangle$ if $\langle a_i \rangle$ and $\langle a_j \rangle$ occur as tensor factors of the same member of At_d. Clearly $\gamma^{i,j}$ and $\gamma^{j,i}$ are inverses.

$$\begin{tikzcd}
\mathsf{At}_{\mathcal{H}_i} \arrow[r, shift left, leftarrow, "\mu^i"] \arrow[r, shift right, "\nu^i"'] \arrow[rrrr, bend left, dashed, "\gamma^{i,j}"'] & \mathsf{At}^i \arrow[r, shift left, leftarrow, "c^i"] \arrow[r, shift right, "d^i"'] & \mathsf{At}_d \arrow[r, shift left, "c^j"] \arrow[r, shift right, leftarrow, "d^j"'] & \mathsf{At}^j \arrow[r, shift left, "\mu^j"] \arrow[r, shift right, leftarrow, "\nu^j"'] & \mathsf{At}_{\mathcal{H}_j} \arrow[llll, bend right, dashed, "\gamma^{j,i}"']
\end{tikzcd}$$

Definition 5.5. *For a Hilbert space $\mathcal{H}$, its* conjugate space $\overline{H}$ *is the same abelian group as $\mathcal{H}$ with scalar multiplication $\lambda \cdot v$ being the scalar multiple by the conjugate $\overline{\lambda}v$ taken in $\mathcal{H}$, and with inner product $[u,v]$ in $\overline{H}$ being the conjugate $\overline{\langle u, v\rangle}$ of the inner product in $\mathcal{H}$.*

It is well known that $\overline{\mathcal{H}}$ is a Hilbert space. A map $U : \mathcal{H} \to \mathcal{K}$ is *anti-unitary* if when considered as a map $U : \mathcal{H} \to \overline{\mathcal{K}}$ it is unitary. We will make use of the following result of Uhlhorn [10] that extends Wigner's theorem.

Theorem 5.6. *For Hilbert spaces $\mathcal{H}, \mathcal{K}$ of dimension ≥ 3, each isomorphism $\phi : \mathsf{At}_{\mathcal{H}} \to \mathsf{At}_{\mathcal{K}}$ lifts to a unique unitary or anti-unitary map $\hat{\phi} : \mathcal{H} \to \mathcal{K}$.*

The dimension of a Hilbert space is the cardinality of a maximal pairwise orthogonal set in the orthogonality space formed from its atoms. Since the orthogonality spaces At^i are isomorphic, the Hilbert spaces $\mathcal{H}_i$ all have the same cardinality. To make use of Ulhorn's theorem, for the remainder of this section we assume that one, hence all, of the Hilbert spaces $\mathcal{H}_i$ has dimension greater than 2.

Definition 5.7. *Let $\hat{\gamma}^i : \mathcal{H}_i \to \mathcal{H}_1$ be the unique unitary or anti-unitary map lifting $\gamma^{i,1}$.*

For a Hilbert space $\mathcal{H}$, let $\mathcal{H}^+ = \mathcal{H}$ and $\mathcal{H}^- = \overline{\mathcal{H}}$ be the conjugate space. Note that $\mathcal{C}(\mathcal{H}) = \mathcal{C}(\overline{\mathcal{H}})$. Let $\alpha \in \{+,-\}^n$ be given by $\alpha_i = +$ if $\hat{\gamma}^i$ is unitary and $\alpha_i = -$ if $\hat{\gamma}^i$ is anti-unitary. Then $\hat{\gamma}^i : \mathcal{H}_i \to \mathcal{H}_1^{\alpha_i}$ is unitary for each $i \leq n$. As with any unitary (or anti-unitary) map between Hilbert spaces, $\hat{\gamma}^i$ induces an isomorphism between their OMLs of closed subspaces via direct images. Further, $\hat{\gamma}^i\langle a_i\rangle = \gamma^{i,1}\langle a_i\rangle$ since $\hat{\gamma}^i$ lifts $\gamma^{i,1}$.

Definition 5.8. *Let $\hat{\gamma} : \bigotimes_1^n \mathcal{H}_i \to \bigotimes_i^n \mathcal{H}_1^{\alpha_i}$ be $\hat{\gamma}^1 \otimes \cdots \otimes \hat{\gamma}^n$.*

Since $\hat{\gamma}$ is unitary, it indices an OML-isomorphism from $L = \mathcal{C}(\bigotimes_1^n \mathcal{H}_i)$ to $L^* = \mathcal{C}(\bigotimes_1^n \mathcal{H}_1^{\alpha_i})$ via direct images and we denote this isomorphism of OMLs also by $\hat{\gamma}$. Since L carries the structure of a strong n-dimensional quantum cylindric algebra, this can be moved across the isomorphism to provide L^* with such structure.

Definition 5.9. *Let c_i^* and $d_{i,j}^*$ be the operations induced on L^* by $\hat{\gamma}$.*

Since $c_i x$ is the least element in $Q_i = \mathcal{H}_i \otimes \mathcal{C}(\bigotimes_{j\neq i} \mathcal{H}_j)$ lying above x and $\hat{\gamma}$ takes Q_i to $Q_i^* = \mathcal{H}_1^{\alpha_i} \otimes \mathcal{C}(\bigotimes_{j\neq i} \mathcal{H}_1^{\alpha_j})$, moving c_i across $\hat{\gamma}$ for $i \leq n$ provides the diagonal-free quantum cylindric algebra structure associated to $\bigotimes_1^n \mathcal{H}_1^{\alpha_i}$. To treat the derived diagonal structure on L^* we use the following.

Lemma 5.10. *The image At_d^* of At_d under $\hat{\gamma}$ is $\{\langle a\rangle \otimes \cdots \otimes \langle a\rangle : \langle a\rangle \in \mathsf{At}_{\mathcal{H}_1}\}$.*

Proof. Let $x = \langle a_1 \rangle \otimes \cdots \otimes \langle a_n \rangle$ belong to At_d. Then the image of x under $\hat{\gamma}$ is $\hat{\gamma}^1 \langle a_1 \rangle \otimes \cdots \otimes \hat{\gamma}^n \langle a_n \rangle$. As noted above, $\hat{\gamma}^i \langle a_i \rangle = \gamma^{i,1} \langle a_i \rangle$ for each $i \leq n$. But $\langle a_1 \rangle$ and $\langle a_i \rangle$ occur as tensor factors of the some element of At_d, namely x, for each $i \leq n$. So $\gamma^{i,1} \langle a_i \rangle = \langle a_1 \rangle$ for each $i \leq n$. So $\hat{\gamma} x = \langle a_1 \rangle \otimes \cdots \otimes \langle a_1 \rangle$. That each $\langle a \rangle \otimes \cdots \otimes \langle a \rangle$ is in the image of At_d follows as each $\langle a \rangle \in \mathsf{At}_{\mathcal{H}_1}$ occurs as a tensor factor of some element of At_d. □

Lemma 5.11. $d^*_{i,j} = \bigvee \{ \langle a \rangle \otimes \langle a \rangle \otimes \bigotimes_{k \neq i,j} \mathcal{H}_1^{\alpha_k} : \langle a \rangle \in \mathsf{At}_{\mathcal{H}_1} \}$.

Proof. Since L is strong, condition (S$_6$) gives $d_{i,j} = \bigvee \{ c^i x \wedge c^j x : x \in \mathsf{At}_d \}$. Since $d^*_{i,j} = \hat{\gamma} d_{i,j}$, we have $d^*_{i,j} = \bigvee \{ (c^i)^* y \wedge (c^j)^* y : y \in \mathsf{At}^*_d \}$. We noted that the induced operations c^*_i are those associated with the diagonal-free quantum cylindric set algebra $\bigotimes_1^n \mathcal{H}_1^{\alpha_i}$. Since c^i is the composite of all c_j for $j \neq i$, we have that $(c^i)^*$ is the composite of the c^*_j for $j \neq i$. So $(c^i)^* y$ is the least element of $(Q^i)^* = \mathcal{C}(\mathcal{H}_1^{\alpha_i}) \otimes \bigotimes_{j \neq i} \mathcal{H}_1^{\alpha_j}$ above y. From the description of the elements $y \in \mathsf{At}^*_d$ of Lemma 5.11 the elements obtained as $(c^i)^* y \wedge (c^j)^* y$ are those of the form $\langle a \rangle \otimes \langle a \rangle \otimes \bigotimes_{k \neq i,j} \mathcal{H}_1^{\alpha_k}$ for $\langle a \rangle \in \mathsf{At}_{\mathcal{H}_1}$. □

Lemma 5.12. *If $\alpha_i \neq \alpha_j$, then $d^*_{i,j} = 1$.*

Proof. Let S be the set of all elements of the form $a \otimes a \otimes v$ where $a \in \mathcal{H}_1$ and $v \in \bigotimes_{k \neq i,j} \mathcal{H}_1^{\alpha_k}$. Throughout we use the convention that the first listed factor in a tensor is for $\mathcal{H}_1^{\alpha_i}$, the second is for $\mathcal{H}_1^{\alpha_j}$ and the third is for $\bigotimes_{k \neq i,j} \mathcal{H}_1^{\alpha_k}$. We will use $\mathcal{H}$ for $\bigotimes_1^n \mathcal{H}_1^{\alpha_i}$. By Lemma 5.11, it is enough to show that the closed subspace $\langle S \rangle$ generated by S is $\mathcal{H}$.

Let $(e_p)_P$ be an ONB of $\mathcal{H}_1$. This same set is an ONB of $\mathcal{H}_1^{\alpha_i}$ for each $i \leq n$. So the $e_{p_1} \otimes \cdots \otimes e_{p_n}$ where $p_1, \ldots, p_n \in P$ are an ONB of $\mathcal{H}$. Therefore, our result will follow if we show that for each $p, q \in P$ and each $v \in \bigotimes_{k \neq i,j} \mathcal{H}_1^{\alpha_k}$ that $e_p \otimes e_q \otimes v$ belongs to $\langle S \rangle$.

Fix v. Note first that from the definition of S

$$e_p \otimes e_p \otimes v \in S \quad \text{for each } p \in P \tag{1}$$

For $p, q \in P$, note that

$$(e_p + e_q) \otimes (e_p + e_q) \otimes v = e_p \otimes e_p \otimes v + e_p \otimes e_q \otimes v + e_q \otimes e_p \otimes v + e_q \otimes e_q \otimes v$$

Then, making use of (1) we have

$$e_p \otimes e_q \otimes v + e_q \otimes e_p \otimes v \in \langle S \rangle \quad \text{for each } p, q \in P \tag{2}$$

For the final step, we use $\cdot_i$, $\cdot_j$ and $\cdot_{i,j}$ for scalar multiplication in $\mathcal{H}_1^{\alpha_i}$, $\mathcal{H}_1^{\alpha_j}$ and $\mathcal{H}_1^{\alpha_i} \otimes \mathcal{H}_1^{\alpha_j}$ respectively. We use $\cdot$ for scalar multiplication in $\mathcal{H}$. Recall that the first listed factor is the i^{th} and the second the j^{th}. Also, as we have assumed that $\alpha_i \neq \alpha_j$ we have that $\lambda \cdot_j a = \overline{\lambda} \cdot_i a$. Note

$$\begin{aligned}(e_p + i \cdot_i e_q) \otimes (e_p + i \cdot_i e_q) \otimes v \;\; &= \;\; e_p \otimes e_p \otimes v + (i \cdot_i e_q) \otimes (i \cdot_i e_q) \otimes v \\ &\quad + e_p \otimes (i \cdot_i e_q) \otimes v + (i \cdot_i e_q) \otimes e_p \otimes v\end{aligned}$$

Since the left side of this equation and the first two terms of the right side of the equation are of the form $a \otimes a \otimes v$, by (1) they belong to S. So the sum of the final two terms of the right side of the equation belongs to $\langle S \rangle$. But

$$\begin{aligned}&e_p \otimes (i \cdot_i e_q) \otimes v + (i \cdot_i e_q) \otimes e_p \otimes v \\ &\qquad = e_p \otimes (-i \cdot_j e_q) \otimes v + (i \cdot_i e_q) \otimes e_p \otimes v \\ &\qquad = [i \cdot_{i,j} (e_p \otimes (-e_q))] \otimes v + [i \cdot_{i,j} (e_q \otimes e_p)] \otimes v \\ &\qquad = i \cdot [e_p \otimes (-e_q) \otimes v + e_q \otimes e_p \otimes v]\end{aligned}$$

Since the element in this equation belongs to $\langle S \rangle$, so does any scalar multiple of it, giving

$$e_p \otimes (-e_q) \otimes v + e_q \otimes e_p \otimes v \in \langle S \rangle \quad \text{for each } p, q \in P \tag{3}$$

Comparing (2) and (3) gives $e_p \otimes e_q \otimes v \in \langle S \rangle$ as required. □

Theorem 5.13. *Let L be a strong n-dimensional quantum cylindric algebra. If for some $i \leq n$ the image of c^i has at least 3 pairwise orthogonal atoms, then L is isomorphic to an n-dimensional full quantum cylindric set algebra.*

Proof. By Proposition 5.1, the diagonal-free reduct of L is an n-dimensional quantum cylindric set algebra associated with $\bigotimes_1^n \mathcal{H}_i$. The atom set of the image of c^i is At^i and this is isomorphic to the atom set of $\mathcal{C}(\mathcal{H}_i)$. Since some At^i for $i \leq n$ has at least at least 3 pairwise orthogonal elements the assumption enforced after Theorem 5.6 that some $\mathcal{H}_i$ has dimension ≥ 3 holds. So L is isomorphic to L^* whose diagonal-free structure is that associated with $\bigotimes_1^n \mathcal{H}_1^{\alpha_i}$. If we can show that $\alpha_i = +$ for each $i \leq n$, then it follows by Lemmas 3.7 and 5.11 that the diagonal structure of L^* is also that of the n-dimensional full quantum cylindric set algebra associated with $\mathcal{H}_1^{\otimes n}$.

Let P be the set of indices $i \leq n$ with $\alpha_i = +$ and N be those indices with $\alpha_i = -$. By definition, the map $\gamma^{1,1}$ is the identity on $\mathsf{At}_{\mathcal{H}_1}$ so its lift $\hat{\gamma}^1$ is the identity on $\mathcal{H}_1$ and is therefore unitary. So $\alpha_1 = +$. Suppose that N is nonempty. By Lemma 5.12, $d^*_{i,j} = 1$ if α_i and α_j have opposite signs. But under our assumption that N is non-empty, if i, j are such that α_i and α_k have the same sign, there is j so that α_j has opposite sign to α_i, α_k. So $d^*_{i,j} \wedge d^*_{j,k} = 1$. Condition ($\mathrm{C}_8$) gives $d^*_{i,k} = c^*_j(d^*_{i,j} \wedge d^*_{j,k})$. Thus $d^*_{i,k} = 1$. Thus $d^*_{i,j} = 1$ for each i, j, and as the generalized diagonal d^* is the meet of all the diagonals $d^*_{i,k}$, it too is equal to 1. Then by definition of $\mathsf{At}^*_d = \mathsf{At}_u = \{\langle a_1 \rangle \otimes \cdots \otimes \langle a_n \rangle : a_i \in \mathsf{At}_{\mathcal{H}_1}\}$. This easily yields contradictions with (S^d_3) and (S^d_4). For instance, taking a, b

orthogonal in $\mathcal{H}_1$ and setting $x = \langle a\rangle \otimes \cdots \otimes \langle a\rangle$ and $y = \langle b\rangle \otimes \langle a\rangle \otimes \cdots \otimes \langle a\rangle$ we have $x \perp y$ and $(c^2)^* x = (c^2)^* y$.

This contradiction shows that indeed all $\alpha_i = +$ hence L is isomorphic to the quantum cylindric set algebra associated with $\mathcal{H}^{\otimes n}$. $\square$

Remark 5.14. We do not know if an n-dimensional diagonal cylindric set algebra based on $(\mathbb{C}^2)^{\otimes n}$ can have diagonal structure satisfying (S_3^d), (S_4^d), (S_5) and (S_6) that is not given by symmetric tensor products. The question is natural because of its relation to an n-qubit system.

6. Axiomatics

In this final section we make a number of remarks and observations about axiomatics. We begin with similarities and differences between the classical and quantum settings.

Remark 6.1. Both the classical and quantum setting have an assumption that the underlying order structure is of a certain form — a powerset Boolean algebra in the classical case and the OML of subspaces of a Hilbert space in the quantum setting.

Remark 6.2. Both the classical and quantum settings make fundamental use of the set of atoms x for which $x = \bigwedge_1^n c^i x$. In the classical setting this is required to be the set of all atoms. In the quantum setting, this is the set At_u of unentangled atoms.

Remark 6.3. In both the classical and quantum settings, the resulting cylindric algebra is obtained by taking an n-fold "power" of a structure formed from the members of At_u that lie beneath the generalized diagonal d. In the classical setting, this structure is a set X and n-fold power is the set X^n that forms a cylindric set algebra. In the quantum setting, At_d is the orthogonality space of a Hilbert space $\mathcal{H}$ and the n-fold tensor power $\mathcal{H}^{\otimes n}$ yields the quantum cylindric set algebra.

As mentioned in Remark 2.2 a quantum cylindric algebra is a significantly weaker structure than a classical cylindric algebra, so additional axioms are needed for strongness in the quantum setting. We next compare axioms for diagonal-free strongness in the classical and quantum settings.

Remark 6.4. Condition (S_1) is condition (T_1) of a strong cylindric algebra in the classical setting. Condition (S_3) is derived [9, Thm. 1.13] for strong classical cylindric algebras from axiom (T_2). In the classical setting (S_4) is true for atoms x, y in any diagonal-free cylindric algebra. Indeed, $x \leq c^i y$ implies $c^i y \not\leq (c^i x)^\perp$, and as $(c^i x)^\perp$ belongs to Q^i, that $y \not\leq (c^i x)^\perp$, so if y is an atom in a Boolean algebra, $y \leq c^i x$. This final step is not valid for atoms in an OML, so we require an axiom to deal with this. Condition (S_2), which is discussed in greater detail below, has two parts. One part says that certain elements commute, which is automatic in the classical setting. The other part limits the amount of commutativity. This is a purely quantum phenomenon and has no counterpart in the classical setting where all elements commute.

Remark 6.5. The strongness axioms in the classical setting do not directly involve the diagonals. In the presence of strongness, the other axioms of classical cylindric algebras, in particular the axiom related to substitution, provides that the diagonal structure is uniquely determined. This is not the case in the quantum setting. On the diagonal-free full quantum cylindric set algebra associated with $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, we can define diagonals $d_{i,j} = 1$ for each i, j and obtain a quantum cylindric set algebra that satisfies (S_1) - (S_4). Axiom (S_6) explicitly describes diagonals in terms of the cylindric structure and holds also in the classical setting.

In the remainder of this section we consider axiom (S_2) which states that $C(Q^i) = Q_i$. This is a higher-level axiom and we break it into lower-level pieces. For this, we introduce two additional axioms.

Definition 6.6. *For each $p, q \in L$ and $i \leq n$ let*

(S_7) $c^i p = (c^i p \vee c_i q) \wedge (c^i p \vee (c_i q)^\perp)$,

(S_8) $c_i p = \bigvee\{(p \vee c^i q) \wedge (p \vee c^i(q^\perp)) : q \in L\}$.

In the following, we work in the diagonal-free setting since the arguments do not involve diagonals. The results apply to the setting with diagonals replacing (S_3), (S_4) with (S_3^d), (S_4^d).

Proposition 6.7. *Axioms* (S_1), (S_3), (S_4), (S_7), (S_8) *imply* (S_2).

Proof. Since the range of c^i is Q^i and that of c_i is Q_i, Axiom (S_7) says that each member of Q^i commutes with each member of Q_i. This gives that $Q_i \subseteq C(Q^i)$. For the converse, suppose that $p \in C(Q^i)$. Then $(p \vee c^i q) \wedge (p \vee c^i(q^\perp)) = p$ for each $q \in L$. Then Axiom (S_8) gives $c_i p = p$, hence $p \in Q_i$. This shows $C(Q^i) \subseteq Q_i$. □

Proposition 6.8. *Axioms* (S_7), (S_8) *are valid in a diagonal-free n-dimensional full quantum cylindric set algebra L.*

Proof. By Proposition 3.9, (S_2) holds in L, hence (S_7) holds. Let $p, q \in L$. Since $p \leq c_i p$ we have $(p \vee c^i q) \wedge (p \vee (c^i q)^\perp) \leq (c_i p \vee c^i q) \wedge (c_i p \vee (c^i q)^\perp)$. By ($\mathrm{S}_7$) the latter expression equals to $c_i p$. Set $s = \bigvee\{(p \vee c^i q) \wedge (p \vee (c^i q)^\perp) : q \in L\}$. Then $s \leq c_i p$. We must show equality.

For $x \in \mathsf{At}$ consider

$$c_i x \leq \bigvee\{(x \vee c^i q) \wedge (x \vee (c^i q)^\perp) : q \in L\}. \tag{4}$$

Suppose we show that (4) holds for all $x \in \mathsf{At}$. Since c_i is a quantifier, it preserves existing joins. Since P is the join of the atoms beneath it, we then have $c_i p = \bigvee\{c_i x : x \in \mathsf{At} \text{ and } x \leq p\}$. It then follows from (4) that

$$c_i p \leq \bigvee\{(x \vee c^i q) \wedge (x \vee (c^i q)^\perp) : q \in L, x \in \mathsf{At} \text{ and } x \leq p\}.$$

This latter expression is the join of terms, each lying beneath s, so it follows that $c_i p \leq s$. So it is enough to show that (4) holds.

Since (4) involves only c_i and c^i, we may assume that our diagonal-free quantum cylindric set algebra is based on $\mathcal{H} \otimes \mathcal{K}$ and $i = 2$. So c_i becomes $\exists_\mathcal{K}$ and c^i becomes $\exists_\mathcal{H}$. Let $x = \langle v \rangle$ be an atom of $\mathcal{C}(\mathcal{H} \otimes \mathcal{K})$. For $(b_i)_I$ an ONB of $\mathcal{K}$ we express $v = \sum_I v_i \otimes b_i$. Using the description of $\exists_\mathcal{K}$ given after Definition 3.2 we have $c_i x = \langle\{v_i : i \in I\}\rangle \otimes \mathcal{K}$. To show (4) it is enough to show

$$\langle\{v_i : i \in I\}\rangle \otimes \mathcal{K} \subseteq \bigvee_{w \in \mathcal{K}} (\langle v \rangle + \mathcal{H} \otimes \langle w \rangle) \cap (\langle v \rangle + \mathcal{H} \otimes \langle w \rangle^\perp). \tag{5}$$

Let RHS be the right side of the containment given in (5). For each $i \in I$ we claim that $v_i \otimes b_i \in$ RHS. Clearly $v_i \otimes b_i$ belongs to the first term of $(\langle v \rangle + \mathcal{H} \otimes \langle b_i \rangle) \cap (\langle v \rangle + \mathcal{H} \otimes \langle b_i \rangle^\perp)$. But $v = \sum_I v_i \otimes b_i$ belongs to the second term of the intersection, and each $v_j \otimes b_j$ for $j \neq i$ belongs to the second term since $b_j \perp b_i$. So $v_i \otimes b_i$ is also in this second term. Thus $v_i \otimes b_i \in$ RHS.

We next show $v_i \otimes b_j \in$ RHS for each $i, j \in I$. To ease notation we show $v_0 \otimes b_1 \in$ RHS. As a first step, consider

$$(\langle v \rangle + \mathcal{H} \otimes \langle b_0 + b_1 \rangle) \cap (\langle v \rangle + \mathcal{H} \otimes \langle b_0 + b_1 \rangle^\perp).$$

Since $v = \sum_I v_i \otimes b_i$, $v_0 \otimes b_0 + v_0 \otimes b_1$, and $v_1 \otimes b_0 + v_1 \otimes b_1$ belong to the first term, so does $-v_1 \otimes b_0 - v_0 \otimes b_1 + \sum_{i \geq 2} v_i \otimes b_i$. Note that $b_0 - b_1 \perp b_0 + b_1$. So $-v$, $v_0 \otimes b_0 - v_0 \otimes b_1$, and $v_1 \otimes b_0 - v_1 \otimes b_1$ are in the second term. Thus $-v_1 \otimes b_0 - v_0 \otimes b_1 - \sum_{i \geq 2} v_i \otimes b_i$ belongs to the second term. Each $v_i \otimes b_i$ for $i \geq 2$ belongs to this second term since each such b_i is orthogonal to $b_0 + b_1$, and this gives $-v_1 \otimes b_0 - v_0 \otimes b_1 + \sum_{i \geq 2} v_i \otimes b_i$ is also in the second term, hence in both terms, and therefore in RHS. But each $v_i \otimes b_i$ for $i \in I$ is in RHS, and it follows that $v_1 \otimes b_0 + v_0 \otimes b_1 \in$ RHS.

As a second step to showing that $v_0 \otimes b_1 \in$ RHS consider

$$(\langle v \rangle + \mathcal{H} \otimes \langle b_0 - ib_1 \rangle) \cap (\langle v \rangle + \mathcal{H} \otimes \langle b_0 - ib_1 \rangle^\perp).$$

Using the same argument as above, since $v_0 \otimes b_0 - iv_0 \otimes b_1$ and $iv_1 \otimes b_0 + v_1 \otimes b_1$ are in the first term, so is $-iv_1 \otimes b_0 + iv_0 \otimes b_1 + \sum_{i \geq 2} v_i \otimes b_i$. Note that $b_0 + ib_1 \perp b_0 - ib_1$. So $v = \sum_I v_i \otimes b_i$, $v_0 \otimes b_0 + iv_0 \otimes b_1$ and $iv_1 \otimes b_0 - v_1 \otimes b_1$ are in the second term, hence so is $iv_1 \otimes b_0 - iv_0 \otimes b_1 - \sum_{i \geq 2} v_i \otimes b_i$. As $v_i \otimes b_i$ for $i \geq 2$ is in RHS, $iv_1 \otimes b_0 - iv_0 \otimes b_1 \in$ RHS. Thus $-v_1 \otimes b_0 + v_0 \otimes b_1 \in$ RHS.

From above, $v_1 \otimes b_0 + v_0 \otimes b_1 \in \textsc{rhs}$, and it follows that $v_1 \otimes b_0$ and $v_0 \otimes b_1$ are in RHS as desired.

Since $v_i \otimes b_j \in \textsc{rhs}$ for each $i, j \in I$, the containment (5) follows. $\square$

Combining Propositions 6.7 and 6.8 we have the following.

Theorem 6.9. *In the axiomatization of n-dimensional full quantum cylindric set algebras, both without and with diagonals,* (S_2) *can be replaced with* (S_7) *and* (S_8)*.*

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